Formal Representation of Beliefs Gradation: probabilistic, multi-valued and graded modalities' approaches

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Abstract. The aim of the article is to provide a comprehensive survey of formal methods currently widely used to model the gradation of beliefs. Thus, it brings together in an insightful way several frameworks representing probabilistic, multi-valued and graded modalities' approaches. They enable to express, in various manners, uncertainty of agents' beliefs and, even more, the process of persuasion which is in fact related to the phenomenon of uncertainty.

1 Introduction

Although a great deal of interest has focused recently on logics of beliefs, there has been relatively little work on providing formalisms for modeling the gradation of beliefs. The need for such a representation arose together with the need for handling the **uncertainty** management in artificial systems and theoretical computer science. Clearly, there are such situations when the intelligent agents must be able to reason and act using unreliable, incomplete or statistical data.

However, our interest in techniques of representing beliefs' gradation goes even further. Our motivation is to choose the most adequate formalism to reason about the process of **persuasion**. The gradation of beliefs plays an important role in describing persuasiveness of particular persuaders or arguments. Indeed, throughout the process of persuasion the belief-attitudes are not only black or white ("It is *for sure* true", "It is *for sure* false"), but they represent various shades of uncertainty such as "*Maybe* you are right", "I am *almost sure* that this is true", "It *seems* to be false", etc. Simply stated the gradation of beliefs enables to express that different proponents using different means of convincing result in unlike effects with respect to a given audience - the audience may become convinced with a various strength (in a various degree). Consider the following statements: "I chose the lesser of two evils when voting for the party X", "The truth is that I didn't vote for X - I rather voted against the party Y". We may understand these as if the speaker was not absolutely certain whether X's electoral program was good, but he voted for X anyway. That is, the party did not persuade the person in a high degree, however it managed to achieve the higher grade than the other party did. Associating the value of 0 with an attitude of "absolutely for no" and 1 with "absolutely for yes", we shall say that the person is convinced to X in the degree lesser than 0.5, but still higher than the degree assigned to Y's opinions. In fact, in real-life practice the rivals may be so weak that the grade assigned to X's reputation may be pretty low, but you vote for X anyway.

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The important question becomes: how are we going to model the gradation of beliefs? Despite applying various mathematical or logical tools, this task is still viewed as very difficult and unsatisfactorily accomplished from the **formal** point of view. Different authors have argued for the appropriateness of different formalisms to capture shades of uncertainty. In the paper, we concentrate on the widespread techniques used to represent gradation of beliefs, namely: probabilistic, multi-valued and graded modalities' representations.

The rest of the paper is organized as follows. In the next section, we discuss the probabilistic approach including the subjective Bayesian method, the certainty factor model, the Dempster-Shafer theory and the probability modal logic. In the chapter 3, we review multi-valued system introduced by Jerzy Łoś as well as Multivalued Logic of Knowledge and Time. In the last section, we demonstrate graded modalities' formalism interpreted epistemically by W. van der Hoek and J.J. Meyer

2 Probabilistic approach

Probability seems to be a very intuitive and natural tool to describe gradation of beliefs. The probabilistic representation has been developed within two trends. The first "camp" adapts classical mathematical theory of probability quite literally. The only difference is that the probability gains the subjective interpretation according to which it is a mode of one's judgment. The second trend combines the probability theory with Kripke-style semantics. Roughly speaking, the basic idea is to place probability measure on the set of possible worlds. In the next subsections, we describe these two trends in detail.

2.1 Theory of subjective probability

The first probabilistic trend has been strongly developed since the 1970s in the research of uncertainty management in expert systems (artificial intelligence). Three main theories were elaborated at that time: the subjective Bayesian method, the certainty factor model and the Dempster-Shafer theory. In the subjective probability framework, uncertainty is reduced more or less directly to mathematical probability. That is, we think of probability as of a bet which an individual makes with respect to a given opinion. For example, you may give odds 1:9 for the opinion that life on Mars exists risking 1 dollar to gain 9 [15]. Then, it corresponds to probability of $\frac{1}{10}$ which can be understood as a degree of your belief about life on Mars.

The subjective Bayesian method The method applies Bayes' theorem in the process of statistical inference in which an evidence is used to update or to newly infer the probability that a hypothesis may be true.[3] More formally speaking, the idea is to find conditional probability of a *hypothesis* h given that an *evidence* e has been observed. Say that you are interested how probable is the event that a particular patient suffers from influenza (h_{flu}) assuming that he shows the symptoms of cough (e_{cough}). For many diseases, a relative frequency of the disease in a specific population, i.e. a **prior probability**, is known. Say that both of the prior probabilities are available, i.e. probabilities of the hypothesis $P(h_{flu})$ and the evidence $P(e_{cough})$. What we want to find is a conditional probability of the hypothesis given evidence, i.e. a **posterior probability** $P(h|e)$. In most of the cases of real-life practice, those probabilities are not available. However, it is relatively easy to find a conditional probability of a type $P(e|h)$. In our example, $P(e_{cough}|h_{flu})$ is the probability that the patient who suffers from flu has cough. Using Bayes' theorem we can now compute a posterior probability that the patient showing signs of a cough has an influenza:

$$P(h_{flu}|e_{cough}) = \frac{P(e_{cough}|h_{flu}) \cdot P(h_{flu})}{P(e_{cough})}.$$

Assuming that the events we consider are mutually exclusive and exhaustive, we may reformulate Bayes' theorem to the more general version:

[3] Bayes' theorem is valid not only in subjective but in all interpretations of probability.

$$P(h_i|e_1,\ldots,e_m) = \frac{P(e_1|h_i)\ldots P(e_m|h_i)\cdot P(h_i)}{\sum_{j=1,\ldots,n} P(h_j)\cdot P(e_1|h_j)\cdot\ldots\cdot P(e_m|h_j)} \text{ for } i=1,\ldots,n.$$

That allows to reason about the probability of a particular disease (h_i) from a set of possible diseases $\{h_1,\ldots,h_n\}$, given a set of observed symptoms $\{e_1,\ldots,e_m\}$. In this manner, the Bayes' theorem enables to revise (graded) beliefs about a given hypothesis in light of new evidences.

The Bayesian technique has become an active topic of investigation for researchers in such diverse fields as mathematics, economics or artificial intelligence. In the 1970s, the so-called **subjective Bayesian method** was introduced for the purpose of handling uncertain information in the expert system PROSPECTOR designed by R.O. Duda, P.E. Hart and N.J. Nilsson [7]. To avoid problems encountered in applying probability theory to expert systems in a straightforward manner, the notion of probability was replaced by the notions of odds and likelihood (see e.g. [7] or [19] for more details).

The certainty factor model The second technique was constructed in the 1970s by E.H. Shortliffe and B.G. Buchanan for dealing with uncertain information in the expert system MYCIN (see e.g. [23]). Although the validity of this technique has been challenged, it has enjoyed widespread use mainly due to its computational simplicity and intuitive formulation.

The basic idea is to assign a numerical weight to the consequent of each rule of a form: *if evidence then hypothesis*, in a rule-based system. Interestingly, Shortliffe and Buchanan argue that $P(h|e) = 1 - P(\overline{h}|e)$ (where $\overline{h}$ is the complement of h) does not correspond to the real-life practice of how experts assign values of uncertainty to the various hypotheses. The authors refer to the observation that an expert is often unwilling to accept that an evidence which supports a given hypothesis to a degree d refutes it to a degree of $1-d$ at the same time. As a result the numerical weight, they associate with a rule, is not a probability but a **certainty factor** which is a difference of two measures (each assigned a value from 0 to 1): the **measure of belief** showing the degree to which an evidence increases the belief in a particular hypothesis, and the **measure of disbelief** expressing the degree to which an evidence decreases the belief in a hypothesis. The value of certainty factor belongs to the interval $[-1,1]$ and is computed as follows: $CF = MB - MD$. The certainty factors are then combined to infer the overall certainty associated with the hypotheses on a basis of the known evidences.

The Dempster-Schafer theory (Evidence theory) The theory was introduced by Arthur Dempster in the 1960s [5, 6], and further extended by Glenn Schafer in the 1970s [22]. It was applied to model inexact reasoning in expert systems in 1980s. In this approach the probability theory is modified in a highest degree mainly by associating the probability (uncertainty) with sets of hypotheses, but not requiring assignment to the individual hypotheses. Interestingly, the motivation was to distinguish uncertainty from ignorance which resulted from incompleteness of information. The serious disadvantage of the Evidence theory is its computational complexity.

The basic notion is a **probability assignment** m defined for subsets of a set of hypotheses Θ (called the frame of discernment). For $x \subset \Theta$ the function m assigns zero to the empty set, the value of $[0,1]$ to each x, and $\sum_{x\subset\Theta} m(x) = 1$. Let us pause here for a moment to give some intuitions. Say that in our highly simplified example, you consider three hypotheses which are to explain the symptoms of cough: influenza, virus infection and allergy (i.e. $\Theta = \{flu, virus, allergy\}$). Assume that you believe with the uncertainty, say, 0.8 that the patient suffers rather from one of the infections, i.e. $m(\{flu, virus\}) = 0.8$, however you are unable to distinguish which of these two diagnoses show a real cause of cough. As we noted above, the standard probability theory would demand assigning the probability to each individual hypothesis - in particular to the diagnosis of influenza itself. When there is no information giving priority to one of these two hypotheses, each would be assigned probability of 0.4. In Dempster-Schafer theory that needn't be done. In this way, the difference between uncertainty and ignorance is highlighted.

The probability m expresses the belief assigned only to a set of hypotheses x itself. However, the total belief in x shall depend also on the beliefs in subsets of x. Thus, the cumulative graded belief is defined in terms of a **belief function** $Bel : 2^\Theta \longrightarrow [0,1]$ such that $Bel(x) = \sum_{y\subset x} m(y)$

for each $x \subset \Theta$. The belief function has an interesting property: $Bel(x) + Bel(\overline{x}) \leq 1$. This means that the sum of the degrees of beliefs in the set of hypotheses and the set's complement may be lesser than 1.

These three concepts are successive extensions of mathematical probability theory. As a result they "inherit" **well-elaborated formal apparatus**. On the other hand, they share some properties which may be viewed as **serious restrictions** when applying them for computing degrees of beliefs in practice. First of all, they assume the *independence of evidences*. Clearly, such an assumption is inconvenient since in real-life we often deal with events which influence one another like e.g. a cough and a headache. The second assumption is the *exhaustiveness* and the *mutual exclusiveness* of the elements of the subsets of hypotheses. This condition is also difficult to accomplish in real-life practice. Designing an expert system, it is hard to collect all the hypotheses which possibly explains the symptoms observed, such as all the diagnoses for signs of cough. Finally, these approaches do not provide a *calculus* with thoroughly elaborated syntax and semantics, although it is relatively easy to do. In this manner, the belief operator is not explicitly expressed in the language. As a result, we are unable to distinguish the objective statements referring to the reality from the subjective ones concerning beliefs about the reality. This means that we cannot say that an agent is convinced of a thesis which is actually false.

2.2 Probabilistic modal logic

In the 1980s, the second trend started to be developed. AI researchers, interested in modeling knowledge and formalizing reasoning methods for distributed intelligent systems, turned their attention to epistemic and doxastic modal logics. To extend this approach to handling the uncertainty in the systems of agents, the modal logic was combined with probability theory. The basic idea is to characterize the grades of beliefs as the **probability on possible worlds** which is derived from the **statistical data** at our disposal. To get some intuition, consider the example given in [14]. Say that you have a statistical information that more than 60% of birds fly. It may be understood that the probability that a randomly chosen bird flies is greater than 0.6. Further, assume that Tweety is a bird. From these two pieces of data, you can infer how probable it is for Tweety to fly - you shall believe it with the degree of uncertainty greater than 0.6.

We characterize probabilistic modal logic basing on the **system** AX_{MEAS} introduced by Ronald Fagin, Joseph Y. Halpern and Nimrod Meggido in [9] and further extended by the first two authors in [8].[4] They take inspiration from Bacchus' statistical logic [1] (which led Halpern to construct the first variant of probability logic [14]) and Nilsson's probability logic [21] (AX_{MEAS} is its formalization). To start with we demonstrate syntax and semantics of this logic. Let $\{1, \ldots, n\}$ be a set of *agents* and V_0 a fixed infinite set of *propositional variables*. The set of *propositional formulas* is the closure of V_0 under the boolean operations: $\neg$ (negation, "not"), $\wedge$ (conjunction, "and"). The formula *true* is defined to be an abbreviation for the formula $p \vee \neg p$ where p is a fixed propositional variable. A basic **probability formula** is an expression: $w_i(\alpha) \geq b$ where b is a rational number and $i \in \{1, \ldots, n\}$. This formula expresses agent i's graded belief in α and intuitively means "according to i, formula α holds with probability at least b". Moreover, some useful abbreviations are introduced, e.g. $w_i(\alpha) = b$ for $(w_i(\alpha) \geq b) \wedge (w_i(\alpha) \leq b)$. Thus, a fact that a degree of i's belief about Tweety's ability to fly is greater than 0.6 may be expressed in the language of AX_{MEAS} in the following way:

$w_i(p) > 0.6$ where w_i stays for "probability according to agent i" and p for "Tweety flies".[5]

The formulas are interpreted in the **probability structure** $M = (S, v, P)$ where S is a set of states (or possible worlds), v is a function which assigns to every state a valuation of

[4] For ease of overview we choose the elements of the system that emphasize only those aspects which are important for modeling beliefs' gradation. Moreover, we slightly modify the notation to standardize the presentation throughout the paper.

[5] Sometimes, they do not make a straightforward connection between probability and degrees of beliefs. However, some of the articles (e.g. [14]) show that they identify these two notions.

propositional variables $v : S \longrightarrow \{\mathbf{0}, \mathbf{1}\}^{V_0}$ and P is a *probability assignment* which assigns to each agent $i \in \{1, \ldots, n\}$ and state $s \in S$ a *probability space* $P_{i,s} = (S_{i,s}, \chi_{i,s}, \mu_{i,s})$ where

- $S_{i,s} \subseteq S$ (called *sample space*),
- $\chi_{i,s}$ is a σ-algebra of subsets of $S_{i,s}$ (called *measurable sets*), i.e., a set of subsets of $S_{i,s}$ containing the empty set and closed under complementation and countable union,
- $\mu_{i,s}$ is a **probability function** on the measurable sets $\mu_{i,s} : \chi_{i,s} \longrightarrow [0, 1]$, i.e. $\mu_{i,s}$ is a mapping from $\chi_{i,s}$ to the real interval $[0, 1]$.[6]

Intuitively, the probability space $P_{i,s}$ shows agent i's probabilities on events, given that the state is s. We associate a truth value with each formula α, writing $M, s \models \alpha$ if the value $\mathbf{1}$ is associated with α by a state s of a model M. Assuming that $M, s \models \alpha$ is inductively defined, we may associate with each formula α the set of the states in which the formula is true and which belongs to $S_{i,s}$, i.e.

$$S_{i,s}(\alpha) = \{s' \in S_{i,s} : M, s' \models \alpha\}.$$

Given a probability structure M and a state s, we define the **semantics of the probability formulas** of AX_{MEAS} as follows:

$$M, s \models w_i(\alpha) \geq b \text{ iff } \mu_{i,s}(S_{i,s}(\alpha)) \geq b.^7$$

This needs some explanation. According to the Kripke-style semantics, the states (worlds) may represent, roughly speaking, different possible "versions" ("images") of reality. Obviously, only one image is actual, but we are not certain which one is that. In non-probabilistic approach, the individual believes that Tweety flies only when it is true in *every* state she considers as a possible version of the reality. The idea for adding the probability is that a formula can be true only in *some* of such states and the (graded) belief is generated anyway. Thus, "Tweety flies" would hold in some states possible from the i's viewpoint, and not in others. Given a model M and a state s, an agent i believes that Tweety flies with a degree greater than 0.6 when according to i's probability assignment at s, the set of worlds where "Tweety flies" holds (i.e. the set $S_{i,s}(p) = \{s' \in S_{i,s} : M, s' \models p\}$) has the probability measure greater than 0.6. Formally, we shall write: $M, s \models w_i(p) > 0.6$ iff $\mu_{i,s}(S_{i,s}(p)) > 0.6$.

Say that the probability structure is a tuple $M = (S, v, P)$ where $S = \{s, s_1, \ldots, s_{10}\}$ and s represents the real world. Let us associate with an agent i and a state s (by means of an assignment P) the sample space consisting only of s and s_1 with s and s_1 both being measurable and having measure 0.3 and 0.7, respectively. That is, the probability space is $P_{i,s} = (S_{i,s}, \chi_{i,s}, \mu_{i,s})$ such that $S_{i,s} = \{s, s_1\}$, $\chi_{i,s} = \{\emptyset, S_{i,s}, \{s\}, \{s_1\}\}$ and $\mu_{i,s} = \{\langle\emptyset, 0\rangle, \langle S_{i,s}, 1\rangle, \langle\{s\}, 0.3\rangle, \langle\{s_1\}, 0.7\rangle\}$. Further, assume that the sentence "Tweety flies" is false in s and true in s_1, i.e. $v(s)(p) = \mathbf{0}$ and $v(s_1)(p) = \mathbf{1}$. Now, is it a case that i's degree of belief about Tweety's ability to fly is greater than 0.6 in a state s? We have $S_{i,s}(p) = \{s' \in S_{i,s} : M, s' \models p\} = \{s_1\}$. The measure of this set is greater than 0.6 since it equals to 0.7. Thus, we have $\mu_{i,s}(S_{i,s}(p)) > 0.6$, so $M, s \models w_i(p) > 0.6$.

Now it is time to give **some comments**. First of all, observe that an *accessibility relation* assumed in the standard doxastic semantics is here replaced with a *probability measure.* As a result, this logic is capable of modeling the situations in which an agent considers some worlds to be more likely than others. Imagine two players Kasia and Magda, each drawing one card from a stack of three cards {Ace, King, the Ten}. Assume that Kasia holds King and she tries to guess what card Magda has. She imagines two versions of the reality - in s_1 Magda holds the Ten and in s_2 she holds Ace. In standard doxastic semantics, both of these states are accessible by means of Kasia's doxastic relation what corresponds to the described situation that Kasia considers both of the versions possible and having the same chances. The question becomes: how are we going to model the case when Kasia evaluates s_2 as more likely version of the reality since it seems to her that Magda is always lucky at cards? As we mentioned above, the key to accomplish this is

[6] To simplify an overview, we assume here the measurable case of the logic (see chapter 3 of [9] for the more general, nonmeasurable case).

[7] The semantics of propositional formulas is defined in the standard manner.

to replace a doxastic relation with a probability function. In this manner, by labeling the sets of states with different values we can assign greater certainty with the state s_2 (say 0.8) than with s_1 (say 0.2). Thus, in a state s_1 we shall have $\mu_{Kasia,s_1}\{s_2\} > \mu_{Kasia,s_1}\{s_1\}$.

Secondly, from the viewpoint of the adequacy of beliefs' description the probability modal logic proposed in [8] introduces important extension with respect to the prior versions (see e.g. [9]). That is, it relates a probability function to an agent and a state. Associating different measures with each *agent* allows individuals to vary in opinions and relating probability to a *state* allows an individual to have different opinions depending on circumstances. In prior papers, the authors define the probability structure as the tuple $M = (S, \chi, \mu, v)$ where S is a set of states, χ is a σ-algebra of subsets of S, μ is a probability function $\mu : \chi \longrightarrow [0, 1]$, and v is a valuation function. Say that the set $\{s_1\} \subseteq S$ has measure 0.6, i.e., $\mu\{s_1\} = 0.6$. Clearly, in this approach the measure is the same for each agent and each state. That is, since there is only one probability function all individuals must share beliefs of the same degrees (in particular, since $\mu\{s_1\} = 0.6$ for each $i \in \{1, \ldots, n\}$) no matter at what state ($\mu\{s_1\} = 0.6$ for each $s \in S$).

Finally, due to placing probability on the set of worlds, we obtain the *subjective* notion of probability used as gradation of beliefs in contrast to the *objective* probability used as statistical measures. The latter represents various assertions about the objective statistical relationships in the (one) actual world, e.g. "More than 60% of all birds fly", "54% of Poles voted for Lech Kaczynski in 2005". On the other hand, the sentences like "Tweety flies" can be associated only with probability interpreted subjectively. Objectively, i.e. when we assume only one state representing the real world, this sentence holds either with probability 1 or probability 0 since in reality Tweety either does fly or does not (above we assumed that p is objectively false since in the actual world s it does not hold). When an agent i starts to create images of the reality, say s_1, then her subjective belief about Tweety becomes graded. As we showed before $M, s \models w_i(p) = 0.7$. That is, i believes in a degree 0.7 that Tweety flies. Interestingly, observe that according to this approach beliefs may be false (just as it is assumed in the standard doxastic logic). Although i believes in a degree 0.7 that Tweety flies, objectively Tweety does not since p is false in the reality represented by s (recall that $v(s)(p) = \mathbf{0}$). In this sense beliefs are subjective as they may vary from how the reality objectively is.

3 Multi-valued representation

Multi-valued logics seem to be particularly appropriate frameworks for modeling uncertainty of opinions. Although there are so many systems which explore more than two logical values, there is relatively little research on connections between multi-valued formalisms and reasoning about degrees of beliefs. We present two approaches and discuss the pros and cons. Especially, we focus on possibilities of expressing various grades of beliefs of different individuals directly in the language of the logic under consideration.

3.1 The System *L* (Jerzy Łoś)

In this subsection, we show the logic L introduced by Jerzy Łoś in the article [18] from 1948. In the L-**language**, the set of first-order logic formulas is extended by a formula of the form: $Lx\alpha$, where x is an individual variable and α is a formula of the system L. The intended reading of $Lx\alpha$ is: an agent x believes that α.

The interesting issue is the **axiomatization** of the L-system. Its formulation somehow reminds of today's standard way of how beliefs are defined in doxastic modal logic (we mean $KD4$ system). However, L-axioms requires beliefs to have slightly different attributes from what is assumed nowadays. The system L has two inference rules: Modus Ponens and Substitution. It has also the following axioms:

L0 classical propositional tautologies
L1 $Lxp \leftrightarrow \neg Lx\neg p$ (consistency and completeness of beliefs)

L2 $Lx(p \to q) \to (Lxp \to Lxq)$ (belief distribution)
L3 $\forall x Lxp \to p$ (collective infallibility)
L4 $LxLxp \leftrightarrow Lxp$ (infallibility and omniscience with respect to own beliefs)

The properties of beliefs postulated in axioms **L0** and **L2** are the same as in the standard approach. **L0** expresses that individuals believe all tautologies of classical propositional calculus. The axiom **L2** requires the consequence in believing. It corresponds to K-axiom in $KD4$. Remaining axioms differ from what is assumed nowadays. They require more idealized attributes of beliefs. The axiom **L1** expresses beliefs' consistency and completeness, i.e. for every two contradictory sentences, the agent must believe one of them and disbelieve the other. In other words, the implication "$\rightarrow$" says that I cannot accept the sentence and its negation at the same time, and the implication "$\leftarrow$" - that I must have a conviction with respect to everything (that is, positive or negative opinion but not neutral). The **L1**-implication "$\rightarrow$" has its equivalent in the system $KD4$ (i.e. D-axiom). Further, the axiom **L3** assumes collective infallibility. This means that whatever is believed by everyone is true. Finally, the axiom **L4** demands infallibility and omniscience with respect to one's own beliefs. That is, the implication "$\rightarrow$" says that my opinions about my beliefs are true, and the implication "$\leftarrow$" - that I am aware of all my beliefs. The **L4**-implication "$\leftarrow$" corresponds to 4-axiom in the system $KD4$.

We can nicely capture the idea of multi-values in L system considering the **truth-table** defined for two agents a and b (see Table 1). Suppose that they disagree with respect to some matters. As a result, the set of all sentences divides into four classes. The class symbolized as **0** consists of the sentences which both individuals disbelieve. The class $\frac{1}{3}$ includes the sentences which the agent a believes and the agent b does not. The class $\frac{2}{3}$, inversely, includes the sentences which a does not believe and b does believe. The class symbolized by **1** consists of the sentences that both individuals believe.

p	Lap	Lbp
0	**0**	**0**
$\frac{1}{3}$	**1**	**0**
$\frac{2}{3}$	**0**	**1**
1	**1**	**1**

Table 1. Four-valued truth-table for beliefs of two agents (called a and b).

Clearly, in this system it is impossible to describe **gradation** of *individual* beliefs. Talking about agent a's beliefs, we may express nothing but her absolute certainty ("positive" or "negative"). That is, the formula Lap may be assigned just one of two values: **1** (when a believes p) or **0** (when a does not believe p). On the other hand, what is expressed by gradation it is a *type of group's disagreement.* For the group of two individuals, a grade $\frac{1}{3}$ denotes that the first agent believes given sentence and the second does not, and a grade $\frac{2}{3}$ - the opposite way. In other words, when a value for a formula Lap is **1** and for Lbp is **0** then in such a case the formula p is assigned a fraction $\frac{1}{3}$. Interestingly, notice that the degrees cannot be directly expressed in the syntax of L since they are defined just by means of semantics of its formulas.

3.2 Multivalued Logic of Knowledge and Time

Another interesting approach to representing knowledge is multi-valued μK-calculus (mv μK) introduced in [17]. It is a very expressive logic which allows specifying knowledge and time in multi-agent systems. The main aim of [17] is to show a model checking technique which can be used for verifying properties involving multivalued μK-calculus or its subsets. However, we focus only on the part concerning modeling knowledge.

Let us start with describing **syntax and semantics** of mv μK. The semantics is based on the notion of interpreted systems which is commonly used in multi-agent scenarios. It assumes a set of *agents* $Agt = \{1, 2, \ldots, n\}$ (in fact, it is a set of names of agents marked by natural numbers). For every agent $i \in Agt$ we assign two sets: Loc_i (a set of *local states* of agent i) and Act_i (a set of *actions* that agent i can execute). As a result, a set of all global states of the system is defined as a subset of the Cartesian product of local states of agents: $S \subseteq Loc_1 \times \cdots \times Loc_n$. Moreover, for every agent $i \in Agt$ we define a function $P_i : Loc_i \to 2^{Act_i}$ which determines which actions can be performed at which local state. Finally, the computation which takes place in the system is modeled by means of a transition function $t : S \times Act \times S \to L$ where $Act \subseteq Act_1 \times \cdots \times Act_n$ is the set of *joint actions* and L is a set of *values* which can be assigned to transitions.[8] Based on this assumptions, a multi-valued transition relation $\mathcal{R} : S \times S \to L \times L$ is defined as follows: $\mathcal{R}((l_1, \ldots, l_n), (l'_1, \ldots, l'_n)) = (v_1, v_2)$ iff there exist actions $a_1 \in P_1(l_1), \ldots, a_n \in P_n(l_n)$ such that $t((l_1, \ldots, l_n), (a_1, \ldots, a_n), (l'_1, \ldots, l'_n)) = v_1$.[9]

Now we are ready to give a **multi-valued model** in which formulas of mv μK are interpreted. Given a set of agents $Agt = \{1, \ldots, n\}$ and a set of propositional variables V_0, by a model we understand a tuple $M = (S, \mathcal{R}, \sim_1, \ldots, \sim_n, V, \mathcal{L})$, where

- S is a finite state of global states of the system,
- $\mathcal{R} : S \times S \to L \times L$ is the multi-valued transition relation on S,
- $\sim_i \subseteq S \times S$ ($i \in Agt$) is an *epistemic accessibility relation* for each agent $i \in Agt$ defined by $s \sim_i s'$ iff $l_i(s') = l_i(s)$, where $l_i : S \to Loc_i$ extracts the local state of agent i from a global state s; observe that $\sim_i$ is an equivalence relation,
- $V : S \times V_0 \to L$ is a *valuation function* for propositional variables,
- $\mathcal{L} = (L, \leq, -)$ is a De Morgan algebra.

For expressing knowledge properties of multi-agent systems in mv μK the well-known **epistemic modalities** K_i, E_Γ, D_Γ are explored. We read them as follows: K_i - "agent i knows", E_Γ - "everybody in group Γ knows", D_Γ - "group knowledge that follows from the individual knowledge of all members of a group Γ". Intuitively, everybody knows a thesis α, written $E_\Gamma\alpha$, iff nobody reckons with an epistemic alternative in which $\neg\alpha$ is true. D_Γ denotes knowledge that is implicitly available within a group Γ: it is the knowledge available to someone who is able to collect the knowledge of all the agents in the group (for instance by sharing the knowledge by means of communication). What is more, in mv μK existential versions of the above operators are used: $\overline{K}_i$, $\overline{E}_\Gamma$, $\overline{D}_\Gamma$. Interestingly, the more sophisticated "common knowledge" operator is defined $C_\Gamma\alpha$. It is true if everyone in Γ knows α, everyone in Γ knows that everyone in Γ knows α, etc.

The semantics of mv μK is given by the function $[\cdot]^M_\rho(s)$, which, for each formula α of mv μK, a model M, a state s in M, and a valuation of the fixed point variables $\rho : Var \to L^S$, returns the value of α at the state s of the model M for the valuation ρ. For propositional and fixed point variables as well as boolean connectives the definition is standard. We quote the semantics only for epistemic modalities:

- $[K_i\alpha]^M_\rho(s) = \bigcap_{\{s' \in S | s \sim_i s'\}}[\alpha]^M_\rho(s')$,
- $[E_\Gamma\alpha]^M_\rho(s) = \bigcap_{i \in \Gamma}[K_i\alpha]^M_\rho(s)$,
- $[D_\Gamma\alpha]^M_\rho(s) = \bigcup_{i \in \Gamma}[K_i\alpha]^M_\rho(s)$,
- $[\overline{K}_i\alpha]^M_\rho(s) = \bigcup_{\{s' \in S | s \sim_i s'\}}[\alpha]^M_\rho(s')$,
- $[\overline{E}_\Gamma\alpha]^M_\rho(s) = \bigcup_{i \in \Gamma}[\overline{K}_i\alpha]^M_\rho(s)$,
- $[\overline{D}_\Gamma\alpha]^M_\rho(s) = \bigcap_{i \in \Gamma}[\overline{K}_i\alpha]^M_\rho(s)$.

Now, we give an example to show more intuitions concerning the operator K. Consider the agent 1 working for the bank, whose task is to decide if a customer, call it agent 2, applying

[8] In [17] it is assumed that $t(s, a, s') = t(s, a', s')$ for any $s, s' \in S$ and $a, a' \in Act$.

[9] For the explanations about v_2 see [17]. We do not mention this since it is not essential for our study.

for a loan is to be granted that loan or not. For this purpose, the agent checks the reliability of the customer in different databases which contain information about bad debtors. Let p be a proposition expressing that "Agent 2 is a good debtor". Furthermore, assume that at state s_1 agent 1 considers as its epistemic alternative four states s_1, s_2, s_3, s_4 (i.e. $s_1 \sim_1 s_j$ for $j = 1, 2, 3, 4$) such that $v(s_1, p) = \frac{3}{4}$, $v(s_2, p) = 1$, $v(s_3, p) = \frac{3}{4}$, $v(s_4, p) = \frac{1}{4}$. The value 1 means that according to databases agent 2 is a good debtor for sure, $\frac{3}{4}$ - he is rather a good debtor, $\frac{1}{4}$ - he is rather a bad debtor. So, the value of the formula $K_1 p$ at the state s_1 is $\bigcap_{s \in \{s_1, s_2, s_3, s_4\}} [p]^M_\rho(s) = \frac{3}{4} \cap 1 \cap \frac{3}{4} \cap \frac{1}{4} = \frac{1}{4}$. As a result, we say that agent 1 knows that agent 2 is rather a bad debtor.

The Multivalued Logic of Knowledge and Time illustrates how a multimodal approach can be adopted in order to reason about different degrees of knowledge of agents in a given system. Although a multivalued formalism provides elegant tools and methods which allow for expressing and analyzing graded knowledge it has some serious **inconvenience**. Note that in fact we can only determine a logical value of a given epistemic formula $K_i\alpha$ but we cannot say with what degree an agent i knows α. Of course those two issues can be identified. That is, we can assume that i knows α with degree d iff the value of $K_i\alpha$ equals d. However, in such a case it is not possible to express nested statements like that: "Agent 1 knows with degree d_1 that agent 2 knows α with degree d_2". Instead of that only the evaluation of the formula $K_1K_2\alpha$ can be given. Needless to explain that it is not the same. Consequently, *nesting of epistemic formulas* with different degrees is not possible.

Furthermore, even if a study is limited to expressions of the form $K_i\alpha$ (where α does not contain epistemic operators) we still do not talk about a degree of knowledge of agent i until a semantic model and an evaluation of this formula are given. It follows from the fact that degrees are not indicated *directly in the syntax* of formulas. Therefore, in some cases specification of properties concerning graded knowledge of agents is very difficult or even impossible.

The most important question which is under our discussion is whether it is possible to adapt mv μK logic to *reason about beliefs*. The attempt to give an answer we made in [2]. To show the main difficulty let us return to the example mentioned above. Observe that agent 1 has four epistemic alternatives. In three of them it stems from databases that the debtor is credible or rather credible and only in one of them the debtor is perceived as unreliable. If we talk about knowledge of agents[10] the conclusion that agent 2 is a bad debtor and the loan should be refused is not surprising. A warning from one source is sufficient to classify the debtor to not granted clients. Now assume that $\sim_i$ is a doxastic relation by means of which we define the semantics of a belief modality B_i, $[B_i\alpha]^M_\rho(s) = \bigcap_{\{s' \in S | s \sim_i s'\}} [\alpha]^M_\rho(s')$. Moreover, let us delete the assumption that $\sim_i$ is reflexive. As a result, all states an agent i considers as its doxastic alternative can mean unreliable sources on the basis of which the agent tries to make a decision. Thereby, if the agent gets positive information from databases s_1, s_2, s_3 and one negative from database s_4 it has grounds for thinking that the result obtained by the database s_4 is false and, as a result, rejecting it. So, the expecting value of the formula $B_i p$ is $\frac{3}{4}$ rater than $\frac{1}{4}$ just as it was fixed by the function $[.]^M_\rho$. For this reason, the adaptation of mv μK to express properties concerning graded beliefs is not straightforward.

4 Graded modalities' formalism

The last framework is called an Epistemic Logic of Graded Modalities (Gr(S5)) and is particulary interesting since it is elaborated in modal epistemic logic itself. The idea of graded operators for modal logic was introduced in 1970s [12, 13, 16] and further studied in 1980s [11, 4, 10]. In the nineties, the epistemic interpretation of graded operators was developed by Wiebe van der Hoek and John Jules Meyer [24–26, 20].

In the formalism we can express that an agent accepts a thesis α although it is conscious of some exceptions to α. Thus the logic with graded modalities allows to deal with types of knowledge

[10] Since the epistemic relation $\sim_i$ is reflexive an agent i knows a thesis T at a state s if T is true at s. In consequence, an agent knows only true facts which follow from true premises. That is, non of the premises can be disregarded.

that are less absolute than in traditional approach. In the standard modal epistemic logic there are considered two kinds of formulas $K_i\alpha$ - α is true in all states accessible via epistemic relation and $M_i\alpha$ (also noted as $\overline{K}_i\alpha$) - α is true in some state accessible via epistemic relation. Thereby, an agent i knows α if α holds in all states it considers as possible. However, in some situations it might be desirable to be able to express that the agent has more confidence in α than in $\neg\alpha$. The logic with graded modalities provides a solution to this problem by adding **quantitative modalities** M^d and K^d ($d \in \mathbb{N}$) enabling to describe the agent's point of view in a more precise manner. Intuitively, we understand the formula $M_i^d\alpha$ as follows: "agent i accepts α iff there are **more** than d accessible states verifying α". In the same spirit, dual formula $K_i^d\alpha$ is true iff at **most** d accessible states refute α. This logic can be employed in multi-agent systems where there are different sources to judge the same proposition and agents have to make a decision on the basis of them. It may happen that results obtained in some sources are false because of faulty sensors or bad calculations. In such cases it is understandable that some data could be refused.

Now, we present formal **syntax and semantics** of Gr(S5). The set of all well-formed expressions of the logic is given by the following Backus-Nauer Form (BNF):

$$\alpha ::= p|\neg\alpha|\alpha \vee \alpha|M_i^d\alpha$$

where p is a propositional variable, d is a natural number and $i \in Agt = \{1, \ldots, n\}$ is a name of an agent.

The formulas are interpreted in **Kripke structure** $M = (S, v, R_1, \ldots, R_n)$, where S is a set of worlds (or states), $v : S \to \{\mathbf{1}, \mathbf{0}\}^{V_0}$ a truth assignment (V_0 is a set of propositions) and R_i ($i = 1, \ldots, n$) a binary relation on S. It is assumed that R_i is an equivalence relation.

For a Kripke structure M the truth of a formula α at $s \in S$ is defined inductively as follows:

- $M, s \models p$ iff $v(s)(p) = \mathbf{1}$ for any $p \in V_0$,
- $M, s \models \neg\alpha$ iff not $M, s \models \alpha$,
- $M, s \models \alpha \vee \beta$ iff $M, s \models \alpha$ or $M, s \models \beta$,
- $M, s \models M_i^d\alpha$ iff $|\{s' \in S|(s, s') \in R_i$ and $M, s' \models \alpha\}| > d$ ($d \in \mathbb{N}$).

Here $|X|$ stands for the cardinality of the set $X \subseteq S$. A formula $K_i^d\alpha$ is an abbreviation for $\neg M_i^d\neg\alpha$. We use also $M!_i^d\alpha$ where $M!_i^0\alpha \Leftrightarrow K_i^0\neg\alpha$, $M!_i^d\alpha \Leftrightarrow M_i^{d-1}\alpha \wedge \neg M_i^d\alpha$, if $d > 0$. From the definition above, it is clear that $M!_i^d$ means "**exactly** d".

The system GR(S5) has two inference rules Modus Ponens and Necessitation:

$$\frac{\alpha, \alpha \to \beta}{\beta}, \qquad \frac{\alpha}{K_0\alpha}$$

and the following schemes of **axioms** (for each $d, d' \in \mathbb{N}$ and $i \in Agt$):

A0 all propositional tautologies
A1 $K_i^0(\alpha \to \beta) \to (K_i^d\alpha \to K_i^d\beta)$
A2 $K_i^d\alpha \to K_i^{d+1}\alpha$
A3 $K_i^0\neg(\alpha \wedge \beta) \to ((M!_i^d\alpha \wedge M!_i^{d'}\beta) \to M!_i^{d+d'}(\alpha \vee \beta))$
A4 $\neg K_i^d\alpha \to K_i^0\neg K_i^d\alpha$
A5 $K_i^0\alpha \to \alpha$

The system with rules Modus Ponens and Necessitation and axioms A0-A3 is the graded modal analogue of the basic modal system **K**. The axiom **A1** is a kind of generalized K-axiom, **A2** and **A3** describe ways to decrease and increase grades in the possibility operators, respectively. **A4** is an analogue of the negative introspection axioms of the modal system **S5**. **A5** expresses that known facts are true what corresponds to T-axiom.

Let us now give **some comments**. First of all, as we noted above the logic GR(S5) is an equivalent of modal **S5** system. Thus its axioms are much more *intuitive* than those of probabilistic modal logic described in section 2.2 and is an elegant tool to deal with various shades of uncertainty. Moreover, due to its similarity to the standard approach it is relatively easy to transform the epistemic logic with graded modalities into a *doxastic* one. It is sufficient to loosen the assumption that the accessibility relation is an equivalence relation and to require that it is serial, transitive, and euclidean. In consequence, an axiomatic system is changed what is precisely presented in our paper [3]. In this manner, that approach can be applied not only to express knowledge but can also be nicely adapted to reason about beliefs.

The other advantage of this formalism is that degrees of knowledge are employed *directly in the language*. As a result, unlike multi-valued approach specification of properties which require *nesting modalities* with different grades is possible. So, we can formally express that an agent 1 considers at most 10 states which refute that an agent 2 leans towards a thesis (i.e. considers at most 2 states which refute a thesis α): $K_1^{10} K_2^2 \alpha$. Note that, the higher the degree of K is, the less certain the knowledge is.

Further, to evaluate a formula $K_i^d \alpha$ we consider the epistemic relation which shows in how many accessible states α holds. Observe that they assume here that all those states are *equally likely*. However, there are situations in which we would like to differ their chances like in the section 2.2 when Kasia and Magda played cards. You can also think that there is a little likelihood of an earthquake somewhere in the center of the Europe while it is highly probable in San Francisco. Formally it could be reflected in values assigned to every couple $(s, s') \in R_i$ $(i \in Agt)$. Unfortunately, in Gr(S5) it is not taken into consideration. Therefore, we cannot express that an agent perceives some states as more probable than the other ones.

Finally, let us assume that an agent 1 is aware of exactly 10 possible situations in which a thesis α holds, i.e. $M!_1^{10}\alpha$. Notice that in such a case it is still not clear whether the agent more believes α than $\neg\alpha$ since we do not know how many states he considers as his epistemic alternative. Say that for two states in which α is false, we suppose that the agent accepts the thesis. On the other hand, if there are one hundred states at which α does not hold, the agent should refuse the thesis. Indeed, along with the growth of accessible states a degree of uncertainty about the current state rises. However, it is reflected neither in the formula $M!_1^{10}\alpha$ nor in the formula $M_1^{10}\alpha$. To complete the information we can consider the conjunction $M!_1^{10}\alpha \wedge M!_1^{100}\neg\alpha$ which says in how many states the thesis is true and in how many false. Actually, it does not improve the situation because it is not possible to compare directly in the language the numbers of states in which the thesis holds with the number of all accessible states. This means that there is no way to indicate the *ratio* of the number of states which are considered by the agent and verify a thesis to the number of all states which are considered by this agent. It could be done only in a metalanguage.

5 Conclusions

In the paper we give a succinct presentation of selected formalisms in which it is possible to express different shades of uncertainty by means of gradation. We demonstrate and compare probabilistic, multi-valued and graded modalities' approaches. In our future work, we would like to apply them to reasoning about a process of convincing.

The probabilistic approach is one of the widely analyzed in the literature. In the proposed systems the notion of certainty is often reduced directly to mathematical probability. As a result, their basic assumptions are distant from practice related to beliefs of individuals. So, it is difficult to adapt them in many real-life scenarios.

The multi-valued logics seem to be well suited formalisms for reasoning about degrees of knowledge or beliefs. However, they have a serious gap which results from a fact that the degrees are "hidden" in the semantics and not explored in the syntax of the language. In consequence, specification of properties which requires mixing beliefs' grades of different agents is not possible.

The graded modalities' formalism is analogue of the system for knowledge which is most popular among computer scientists and AI researches (i.e. S5 modal system). Thereby, it has more intuitive axiomatization and nicer technical properties than the other proposals discussed. Furthermore, it allows for nesting epistemic operators with different grades of beliefs. That is why, we find it the most appropriate for our reasons.

The adaptation of the epistemic logic with graded modalities to formalizing the process of persuasion is presented in [3] where the sound and complete deductive system for reasoning about graded beliefs and their change is given. In our earlier work [2] we analyzed the possibility of application of multi-valued logics to a process of convincing. However, during our research we found out that the graded modalities' representation are interesting and worth considering as well, sometimes even more adequate and expressible. This paper is a summary of our study in this field.

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