Non-Archimedean Probabilities and Non-Archimedean Bayesian Networks

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In the paper we consider non-Archimedean fuzziness and probabilities. The idea of non-Archimedean multiple-validities is that (1) the set of values for the vagueness and probability is uncountable infinite and (2) this set is not well-ordered. For the first time the non-Archimedean logical multiple-validity was proposed in [13], [14].

We propose non-Archimedean fuzziness that is defined on an infinite-order class of fuzzy subsets in the framework of infinite-order (ω-order) vagueness. This approach allows to set an ω-order fuzzy logic such that its well-formed formulas have truth values in an interval $[0, 1]$ of hyperreal or hyperrational numbers. On the base of non-Archimedean fuzzy logic we can build also non-Archimedean probability logic. In this paper I propose to define probabilities an algebra of fuzzy subsets. These probabilities are said to be fuzzy ones. Their main originality consists in that some Bayes' formulas do not hold in the general case. In the framework of ω-order vagueness we can construct infinitely hierarchical Bayesian networks. For instance, we can consider i-order variables of Bayesian network as i-tuples of first-order variables and ω-order variables as infinite tuples of first-order variables. Also, for the first time we propose to use infinite-order logical constructions for setting fuzziness and probabilities.

Let us remember that Archimedes' axiom affirms: for any positive real or rational number ε, there exists a positive integer n such that $\varepsilon \geq \frac{1}{n}$ or $n \cdot \varepsilon \geq 1$.

The field that satisfies all properties of $\mathbf{R}$ without Archimedes' axiom is called the field of *hyperreal numbers* and it is denoted by ${}^*\mathbf{R}$. The field that satisfies all properties of $\mathbf{Q}$ without Archimedes' axiom is called the field of *hyperrational numbers* and it is denoted by ${}^*\mathbf{Q}$. By definition of field, if $\varepsilon \in \mathbf{R}$ (resp. $\varepsilon \in \mathbf{Q}$), then $1/\varepsilon \in \mathbf{R}$ (resp. $1/\varepsilon \in \mathbf{Q}$). Therefore ${}^*\mathbf{R}$ and ${}^*\mathbf{Q}$ contain simultaneously infinitesimals and infinitely large integers: for an infinitesimal ε, we have $N = \frac{1}{\varepsilon}$, where N is an infinitely large integer.

In the standard way, probabilities are defined on an algebra of subsets. Recall that an *algebra $\mathcal{A}$ of subsets $A \subset X$* consists of the following: (1) union, intersection, and difference of two subsets of X; (2) $\emptyset$ and X. Then a *finitely additive probability measure* is a nonnegative set function $\mathbf{P}(\cdot)$ defined for sets $A \in \mathcal{A}$ that satisfies the following properties:

1. $\mathbf{P}(A) \geq 0$ for all $A \in \mathcal{A}$,
2. $\mathbf{P}(X) = 1$ and $\mathbf{P}(\emptyset) = 0$,
3. if $A \in \mathcal{A}$ and $B \in \mathcal{A}$ are disjoint, then $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$. In particular $\mathbf{P}(\neg A) = 1 - \mathbf{P}(A)$ for all $A \in \mathcal{A}$.

It is possible also to set probabilities on an algebra $\mathcal{F}^V(X)$ of fuzzy subsets $A \subset X$ that consists of the following: (1) union, intersection, and difference of two *fuzzy* subsets of X; (2) $\emptyset$ and X. In this case a *finitely additive probability measure* is a nonnegative set function $\mathbf{P}(\cdot)$ defined for sets $A \in \mathcal{F}^V(X)$ that runs the set V and satisfies the following properties:

1. $\mathbf{P}(A) \geq 0$ for all $A \in \mathcal{F}^V(X)$,
2. $\mathbf{P}(X) = 1$ and $\mathbf{P}(\emptyset) = 0$,
3. if $A \in \mathcal{F}^V(X)$ and $B \in \mathcal{F}^V(X)$ are disjoint, then $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$.
4. $\mathbf{P}(\neg A) = 1 - \mathbf{P}(A)$ for all $A \in \mathcal{F}^V(X)$,

where 1 is the largest member of V and 0 is the least member of V.

This probability measure is called *fuzzy probability.* The main originality of fuzzy probabilities is that conditions 3, 4 are independent. As a result, in a probability space $\langle X, \mathcal{F}^V(X), \mathbf{P}\rangle$ some Bayes' formulas do not hold in the general case.

A probability space $\langle X, \mathcal{F}^{\mathrm{Q}_V}_\infty(X), \mathbf{P}\rangle$ will say to be non-Archimedean. As we see it is a particular case of fuzzy probability space and non-Archimedean probability measure is a particular case of fuzzy probabilities.

A *non-Archimedean Bayesian network* $\mathcal{N}_\infty$ consists of the following

- V is a set included variables $v^i_1, \ldots, v^i_N$ of various order $i \in \omega$ and variables $v^\infty_1, \ldots, v^\infty_N$ of ω-order.
- A is a union of (1) a set of i-order arc towers ($i \in \omega$), which together with V constitutes an i-order dag $\mathcal{G}_i$ over variables $v^1_1, \ldots, v^1_N$ at the first level, over variables $v^i_1, \ldots, v^i_N$ at the i-th level, etc., and (2) a set of ω-order arc towers, which together with V constitutes an ω-order dag $\mathcal{G}_\infty$ over variables $v^\infty_1, \ldots, v^\infty_N$.
- $\mathcal{P}$ is a set of i-order conditional probabilities $\mathbf{P}_i(v^i_j|\pi_{v^i_j})$ of the all i-order variables v^i_j given their respective i-order parents $\pi_{v^i_j}$ ($i \in \omega$) and of ω-order conditional probabilities $\mathbf{P}_\infty(v^\infty_j|\pi_{v^\infty_j})$ of the all ω-order variables v^∞_j given their respective ω-order parents $\pi_{v^\infty_j}$.

Also we have a multihierarchical (more precisely, infinitely hierarchical) Bayesian network. For instance, we can consider i-order variables as i-tuples of first-order variables and ω-order variables as infinite tuples of first-order variables.

The main idea of non-Archimedean Bayesian networks is that we can define multihierarchical structures and consider joint distributions for different levels $i = 1, 2, \ldots$ Principles of setting an infinite hierarchy of Bayesian networks depend on practical aims.

Non-Archimedean Bayesian networks can be also regarded on the set of formulas of non-Archimedean probability logic.

Non-Archimedean fuzziness and non-Archimedean probabilities were considered as infinite-order ones, e.g. we constructed an appropriate ω-order fuzzy and probability logic that allows to get a non-Archimedean fuzzy class theory and non-Archimedean probability theory. On the base of non-Archimedean probabilities we can define also non-Archimedean Bayesian networks and other practical applications.

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